Area expectation values in quantum area Regge calculus

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Abstract

The Regge calculus generalised to independent area tensor variables is considered. The continuous time limit is found and formal Feynman path integral measure corresponding to the canonical quantisation is written out. The quantum measure in the completely discrete theory is found which possesses the property to lead to the Feynman path integral in the continuous time limit whatever coordinate is chosen as time. This measure can be well defined by passing to the integration over imaginary field variables (area tensors). Averaging with the help of this measure gives finite expectation values for areas.

The standard canonical quantisation prescription requires continuous coordinate which would play the role of time. Therefore this prescription is not defined in the completely discrete theory such as Regge calculus formulation of general relativity (GR). Due to the absence of a unique recipe of how to quantise, a lot of different approaches to discrete quantum gravity is possible. In particular, a large amount of hopes are connected with the spin foam approaches (see [1, 2] for a review). The latter are the 4D generalisations of the 3D Ponzano-Regge model of quantum gravity [3] where the partition function of 3D Regge calculus is taken as a discrete sum of the products of $6j$-symbols corresponding to angular momenta j_i for separate tetrahedra with linklengths $j_i + \frac{1}{2}$. The basis for such a choice is that the exponential of 3D Regge action arises in the asymptotic form of $6j$-symbols at large j_i. Thus, in the spin foam models the partition function is more fundamental object than the action since the latter arises only in the asymptotic form of the former. As for the standard canonical quantisation approaches issuing directly from the action, these are applicable to the theories with discrete space but continuous time and are faced with the complication consisting in the fact that discrete constraints of GR are generally second class (i.e. do not commute). This is usual problem with lattice regularisation in quantum gravity for it breaks diffeomorphism invariance (see [4] for a review). This enforces the researchers working in this field to develop some analogs of canonical quantisation of general relativity for the completely discrete spacetimes [5].

In this letter we quantise in terms of the Feynman path integral a model of the 4D Regge calculus [6] which does not possess the above difficulty, i.e. noncommutativity of the constraints (in the discrete space but continuous time). Moreover, it turns out that the quantum measure for the completely discrete theory does exist which possesses the following property (of equivalence of the different coordinates). If we pass to the limit when any of the coordinates is made continuous one then we can define in the natural way the limit for this measure which turns out to be just the Feynman path integral measure corresponding to the canonical quantisation with this coordinate playing the role of time. The model in question is some modification of the usual Regge calculus, the so-called *area* Regge calculus where area tensors are treated as independent variables. The idea that Regge calculus should be formulated in terms of the areas of the triangles rather than the edge lengths was suggested by Rovelli [7] and further studied in ref [8]. In [8] the area Regge calculus with independent (scalar) areas has been introduced, and our model seems to be another one not reducable to that of [8]. At the same time, both versions of area Regge calculus possess simple equations of motion which state that angle defects vanish. Because of the lack of metric and of the usual geometric interpretation of angle defect, this does not mean flat spacetime. Simple form of the equations of motion is, however, sufficient to ensure closeness of the algebra of constraints (i.e. their being first class) even on the discrete level. Evidently, considering the area variables as independent ones means that the theory acquires additional degrees of freedom. Remarkable, however, is that this leads to simplification of the theory.

Now let us pass to the action of our specific version of area Regge calculus of interest

[6] (in slightly modified notations),

$$S(\pi, \Omega) = \sum_{\sigma^2} |\pi_{\sigma^2}| \arcsin \frac{\pi_{\sigma^2} \circ R_{\sigma^2}(\Omega)}{|\pi_{\sigma^2}|}, \tag{1}$$

where $\pi^{ab}_{\sigma^2}$ are the tensors of the 2-faces σ^2 which in the particular case of the usual link vector formulation (not area modified) should reduce to $\epsilon^{abcd} l_{1c} l_{2d}$, l^a_1, l^a_2 being 4-vectors of some two edges of the triangle σ^2 in the local frame of a certain 4-tetrahedron containing σ^2. In our case $\pi^{ab}_{\sigma^2}$ are independent area tensors. The curvature matrix $R_{\sigma^2}(\Omega)$ is the path-ordered product of SO(4) (in the Euclidean case) matrices $\Omega^{\pm 1}_{\sigma^3}$ living on the 3-faces σ^3 taken along the loop enclosing the given 2-face σ^2. Some further notations are $A \circ B \equiv \frac{1}{2} \mathrm{tr} A^{\mathrm{T}} B$ for any two tensors A, B and $|\pi_{\sigma^2}| = (\pi_{\sigma^2} \circ \pi_{\sigma^2})^{1/2}$ for twice the area of the 2-face σ^2.

The whole procedure of constructing measure includes the following steps:
(i) passing to the continuous time;
(ii) writing out Feynman path integral following from the Hamiltonian form and canonical quantisation;
(iii) finding the measure for the completely discrete theory (including time) which would result in the above path integral in the continuous time limit irrespectively of what coordinate is chosen as time.

It is convenient to consider the case of the Euclidean signature. Once the derivation is made, it turns out that the measure obtained can be well defined by passing to the integration over imaginary contours which looks like the formal change of integration variables $\pi \to -i\pi$,

$$< f(\pi, \Omega) >= \int f(\pi, \Omega) d\mu(\pi, \Omega) \Rightarrow \int f(-i\pi, \Omega) d\mu(-i\pi, \Omega). \tag{2}$$

It is convenient to write out integrals in such the form from the very beginning, what just is done in the present paper. The considered derivation is performed quite analogously to the case of 3D Regge calculus considered by the author in the paper [9]. This is connected with the fact that the 4D area Regge calculus resembles the 3D Regge calculus.

Passing to the continuous time limit means that some set of the triangles, call these timelike ones, have area tensors of the order of $O(dt)$ like

$$\pi_{(i^+ik)} \stackrel{\mathrm{def}}{=} n_{ik} dt. \tag{3}$$

Here i^+ means image in the leave at the moment $t + dt$ of the vertex i taken at the moment t. (The leaves are themselves 3D Regge manifolds.) The notation $(i_1 i_2 \dots i_{n+1})$ means unordered n-simplex with vertices i_1, i_2, $\dots$, i_{n+1} (triangle at $n = 2$) while that without parentheses means ordered simplex. Besides that, the connection matrices living on the 3-tetrahedrons with volume $O(1)$ (spacelike and diagonal ones) can be considered as those responsible for the parallel transport at a distance $O(dt)$ and therefore it is natural to put these being infinitesimally close to unity. In fact, the transport to the next $(t + dt)$ time leave is defined by the sum of contributions from the spacelike

$(iklm)$ and diagonal (differing from $(iklm)$ by occurence of superscript "+" on some of i, k, l, m) tetrahedra inside the 4D prism with bases $(iklm)$ and $(i^+k^+l^+m^+)$. Denote the resulting rotation matrix $1 + h_{(iklm)}dt$. The considered n_{ik} and $h_{(iklm)}$ turn out to be Lagrange multipliers at the constraints in the resulting Lagrangian while area tensors on spacelike triangles $\pi_{(ikl)}$ become dynamical variables. As for the dynamical variables of the type of connection, these live on the timelike tetrahedrons like (i^+ikl). It turns out, however, that dynamical connections live, in fact, on the spacelike triangles. Indeed, consider contribution of a diagonal triangle, e.g. (ik^+l), into action. The leading $O(1)$ contribution into action is due to the connections on the two timelike tetrahedrons sharing this triangle; suppose these prove to be (ik^+kl) and (ik^+l^+l). Then equations of motion for the area tensor of this triangle say that these connections are equal, may be, up to possible inversion, $\Omega_{(ik^+kl)} = \Omega^{\pm 1}_{(ik^+l^+l)}$. This means that timelike connection is a function, in essense, on the set of bases of the 3D prisms where the considered timelike tetrahedrons are contained. Explicit calculation of the continuous time action for the usual (with independent linklengths) Regge calculus in the tetrad-connection representation has been made by the author [10]. Now, in area tensor-connection Regge calculus the problem is, in view of the above discussion, much simpler, and expression for the Lagrangian can be obtained from the result of [10].

$$
\begin{aligned}
L &= L_{\dot{\Omega}} + L_h + L_n, && (4)\\
L_{\dot{\Omega}} &= \sum_{(ikl)} \pi_{(ikl)} \circ \Omega^{\dagger}_{(ikl)} \dot{\Omega}_{(ikl)}, && (5)\\
L_h &= \sum_{(iklm)} h_{(iklm)} \circ \sum_{\text{cycle perm}\, iklm} \varepsilon_{(ikl)m} \Omega^{\delta_{(ikl)m}}_{(ikl)} \pi_{(ikl)} \Omega^{-\delta_{(ikl)m}}_{(ikl)} && (6)\\
&\stackrel{\text{def}}{=} C(h)\\
(\delta &\stackrel{\text{def}}{=} \frac{1+\varepsilon}{2}),\\
L_n &= \sum_{(ik)} n_{(ik)} \circ R_{(ik)} && (7)\\
&\stackrel{\text{def}}{=} R(n)\\
(R_{(ik)} &= \Omega^{\varepsilon_{ikl_n}}_{(ikl_n)} \dots \Omega^{\varepsilon_{ikl_1}}_{(ikl_1)}, \quad \varepsilon_{ikl_j} = -\varepsilon_{(ikl_j)l_{j-1}} = \varepsilon_{(ikl_j)l_{j+1}}).
\end{aligned}
$$

Here $\varepsilon_{(ikl)m}$ is a sign function which put in correspondence +1 or -1 to each pair of tetrahedron $(iklm)$ and triangle (ikl) contained in it. It is specified only by conditions presented in (7). The infinitesimal area tensors enter as $n_{(ik)} = n_{ik} + n_{ki}$. Important property is absence of the 'arcsin' function; this is because equations of motion have the same solution $R = \pm 1$ as if 'arcsin' were omitted.

The eqs. (4) - (7) present the system of the first class constraints C, R and kinetic term analogous to those in 3D case considered in [9] and first suggested for the discrete gravity by Waelbroeck [11]. The difference is, first, in the local group (SO(4) versus SO(3)) and, second, in the different topology in 4 and 3 dimensions (here by 'topology' we mean the scheme of connection of the different vertices); in other respects situation

is similar and, in particular, (Euclidean) Feynman path integral measure reads

$$d\mu = \exp\left(i\!\int\! L_{\dot{\Omega}} dt\right)\delta(C)\delta(R)D\pi\mathcal{D}\Omega, \quad D\pi \stackrel{\rm def}{=} \prod_{(ikl)} d^6\pi_{(ikl)}, \quad \mathcal{D}\Omega \stackrel{\rm def}{=} \prod_{(ikl)} \mathcal{D}\Omega_{(ikl)} \tag{8}$$

where $\mathcal{D}\Omega$ is the Haar measure. Upon raising C, R from δ-functions to exponent with the help of the Lagrange multipliers h, n the $d\mu$ can be rewritten as

$$d\mu = \exp\left(i\!\int\! L dt\right) D\pi D n \mathcal{D}\Omega D h. \tag{9}$$

Important is the problem of fixing the gauge and separating out the volume of the symmetry group generated by constraints. The gauge subgroup generated by C consists of SO(4) rotations (in the different terahedrons) and has finite volume. Therefore there is no need to fix the rotational symmetry. The $R(n)$ generate shifts in the values of area tensors when $\pi_{(ikl)}$ changes by the lateral surface of the 3D prism with the base (ikl) which is algebraic sum of $n_{(ik)}$, $n_{(kl)}$, $n_{(li)}$. (More accurately, the sum of the expressions of the type $\Gamma^\dagger n \Gamma$ where Γ are some products of matrices $\Omega^{\pm 1}$ needed to express n's in the frame of the same tetrahedron where $\pi_{(ikl)}$ is defined.) Evidently, $R(n)$ is an analog of the Hamiltonian constraint in the usual GR which governs the dynamics. Let us fix the symmetry generated by $R(n)$ by fixing area tensors of a certain set of the triangles. The number of the triangles from this set should be the same as the number of independent constraints R. The full number of the constraints R is $N^{(3)}_1$, the number of links in the 3D leave, but $N^{(3)}_0$ of these are consequences of others due to the Bianchi identity which can be written for each vertex; $N^{(3)}_0$ is the number of vertices in the 3D leave. Thus, the number of independent constraints R is $N^{(3)}_1$ - $N^{(3)}_0$. Consider now *the set of x-like* triangles F in the 3D leave where x is some coordinate in the leave. The number of these triangles is $N^{(3)}_1$ - $N^{(3)}_0$ as well. More careful investigation shows that the matrix of the Poisson brackets $\{R, f\}$ is nondegenerate for $f = \{\pi_{(ikl)} - a_{(ikl)} | (ikl) \in F\}$, $a_{(ikl)} = const$ (it's determinant is unity under appropriate boundary conditions on the manifold) and thereby $f = 0$ is admissible gauge condition. By the standard rule of the Faddeev-Popov ansatz for separating out the gauge group volume, it is easy to find that such gauge fixing amounts to simply omitting integrations over $d^6\pi_{(ikl)}$, $(ikl) \in F$.

For subsequent constructing the completely discrete measure respecting the coordinate equivalence it is important to convince ourselves that analogously integration over area tensors of the timelike triangles could be omitted. Indeed, due to conservation in time it is sufficient to impose the constraint R only as initial condition. Therefore integration over Dn giving $\delta(R)$ in other moments of time is not necessary and can be omitted.

As a result, one can use the measure either in the more symmetrical form (9) or in similar form where integration over Dn is omitted or in that one where integration over $d^6\pi_{(ikl)}$, $(ikl) \in F$ is omitted, F being the set of x-like triangles.

Now we are in a position to generalise the canonical quantisation (continuous time) measure to the full discrete measure. First, we can recast the measure into the equivalent

form by inserting integration over variables living on the diagonal triangles. Let, for example, the diagonal triangles (ik^+l) and (ik^+l^+) exist shared by the tetrahedra (ik^+kl), (ik^+l^+l) and (ik^+l^+l), $(i^+ik^+l^+)$, respectively. Add contribution of these triangles to the Lagrangian,

$$\pi_{(ik^+l)} \circ \Omega^\dagger_{(ik^+kl)} \Omega_{(ik^+l^+l)} + \pi_{(ik^+l^+)} \circ \Omega^\dagger_{(ik^+l^+l)} \Omega_{(i^+ik^+l^+)}, \tag{10}$$

and at the same time insert integrations over $d^6\pi_{(ik^+l)}$, $d^6\pi_{(ik^+l^+)}$ and substitute $\mathcal{D}\Omega_{(ikl)}$ by $\mathcal{D}\Omega_{(ik^+kl)}$ $\mathcal{D}\Omega_{(ik^+l^+l)}$ $\mathcal{D}\Omega_{(i^+ik^+l^+)}$. Integrations over $d^6\pi_{(ik^+l)}$, $d^6\pi_{(ik^+l^+)}$ give δ-functions of (antisymmetric parts of) the curvature matrices which can be read off from (10). These δ's are then integrated out and, by properties of invariant Haar measure, the two additional integrations over connections reduce to unity.

Second, it is natural to adopt the following rule of passing from the integration over finite rotations to that over infinitesimal ones,

$$\mathcal{D}\Omega \rightarrow d^6 h, \tag{11}$$

if $\Omega = 1 + hdt$. Then the most symmetrical w.r.t. the different simplices expression for completely discrete measure should take the form

$$d\mathcal{M}_{\mathcal{F}} = \exp\left(i \sum_{(ABC)} \pi_{(ABC)} \circ R_{(ABC)}(\Omega)\right) \prod_{(ABC)\notin\mathcal{F}} d^6\pi_{(ABC)} \prod_{(ABCD)} \mathcal{D}\Omega_{(ABCD)}. \tag{12}$$

Here A, B, C, ... denote vertices of the 4D Regge manifold. Integration is omitted over area tensors of the set of triangles $\mathcal{F}$. Occurrence of this set means that full symmetry w.r.t. the different simplices is not achieved, but a'priori there is an arbitrariness in the choice of $\mathcal{F}$, so we can speak of a kind of spontaneous symmetry breaking. Existence of this set is connected with Bianchi identities which can lead to singularity. Indeed, when integrating over $d^6\pi$ the δ-functions of (antisymmetric part of) the curvature can arise, their arguments being generally not independent just due to the Bianchi identities and possibility to have something like δ-function squared exists. Therefore $\mathcal{F}$ is just the set of those triangles the curvature matrices on which are functions of other curvatures. The number of such triangles is evidently the number of independent Bianchi identities. Since Bianchi identity in the 4D case can be written for the curvatures on all the triangles sharing a given link but these identities for all the links meeting at a given vertex are dependent, the above number is $N^{(4)}_1$ - $N^{(4)}_0$, $N^{(4)}_j$ being the number of simplices of the dimensionality j in the 4D manifold. The triangles of the $\mathcal{F}$ should constitute a surface which passes through all the links. Let us choose any coordinate, denote it t, and consider all the t-like triangles. Their number is just $N^{(4)}_1$ - $N^{(4)}_0$, and one can prove that starting from initial conditions on some t-leave one can successively express the curvatures on these triangles in terms of other curvatures.

Further, the set $\mathcal{F}$ naturally fit to the requirement to yield the canonical quantisation measure in the continuous time limit. In fact, different choices of $\mathcal{F}$ correspond to different choices of gauge fixing in the measure (9). Indeed, suppose we make the coordinate t continuous. Then, if $\mathcal{F}$ is the set of the t-like triangles, the limit of the

measure (12) will be (9) with integration over area tensors Dn on the timelike triangles omitted. The choice of $\mathcal{F}$ being the set of x-like triangles results in the limiting measure (9) with integration over area tensors on the set $F \in \mathcal{F}$ x-like triangles omitted. Finally, the choice for $\mathcal{F}$ being some set of the diagonal xt-like triangles leads to the measure (9) with all area tensor integrations available.

The exponential of the measure constructed containes the terms of the two types: contributions of the t-like triangles (if $\mathcal{F}$ is the set of t-like triangles for some coordinate t) and contributions of the spacelike and diagonal triangles. The former can be cast to the form $n \circ R(R, \Omega)$ where $R(R, \Omega)$ are (rather bulky) expressions for the curvatures on $\mathcal{F}$ in terms of other curvatures and connections which solve the Bianchi identities, but coefficients of them, tensors of the t-like triangles n serve as parameters and can be chosen by hand. The latter are $\pi \circ R$ where, in principle, R can be taken as independent variables. Note that scaling by the imaginary unity (2) refers only to the integration (dummy) variables. The t-like triangle tensors n enter real, and Euclidean expectation values are defined as

$$
\begin{aligned}
< f(\pi, \Omega) > \; &= \int f(-i\pi, \Omega) \exp \left(- \sum_{\substack{t-\text{like} \\ (ABC)}} n_{(ABC)} \circ R_{(ABC)}(\Omega) \right) \\
& \exp \left(i \sum_{\substack{\text{not} \\ t-\text{like} \\ (ABC)}} \pi_{(ABC)} \circ R_{(ABC)}(\Omega) \right) \prod_{\substack{\text{not} \\ t-\text{like} \\ (ABC)}} d^6 \pi_{(ABC)} \prod_{(ABCD)} \mathcal{D}\Omega_{(ABCD)}. \qquad (13)
\end{aligned}
$$

Using possibility to choose tensors n by hand, take these negligibly small. If the function to be averaged does not depend on the connections, it is easy to see, with taking into account invariance property of the Haar measure, that then the measure splits into the product of the measures over separate triangles of the type

$$
\exp{(i\pi \circ R)} d^6 \pi \mathcal{D} R. \qquad (14)
$$

In turn, we can use the group property SO(4) = SO(3) $\times$ SO(3) to expand variables (π and generator of R) into self- and antiselfdual parts, in particular, π is mapped into 3-vectors ${}^{+}\boldsymbol{\pi}$, ${}^{-}\boldsymbol{\pi}$. Thereby the measure (14) is represented as the product of the two measures each of which being copy of the measure which appears in the 3D model [9]. In that paper it has been found that the recipe (2) which means now 3D analog of (13) indeed defines a positive measure if one neglects links given by hand (analogs of n). Besides that, expectation value of any power $k >$ -1 of the link vector squared $\mathbf{l}^{2k}$ turns out to exist,

$$
< \mathbf{l}^{2k} > = \frac{4^{-k} \Gamma(2k+2)^2}{\Gamma(k+2)\Gamma(k+1)} \qquad (15)
$$

(in Plank units) which extends to any function

$$
< g(\mathbf{l}) > = \int \frac{do_l}{4\pi} \int_0^\infty g(\mathbf{l}) \nu(l) dl,
$$

$$\nu(l) = \frac{2l}{\pi} \int_0^{\pi} \exp\left(-\frac{l}{\sin\varphi}\right) d\varphi. \tag{16}$$

Now one should substitute here ${}^{\pm}\boldsymbol{\pi}$ instead of $\mathbf{l}$ in order to calculate expectation values of any function of area tensor, e.g. the area itself

$$|\pi|^2 = ({}^{+}\boldsymbol{\pi})^2 + ({}^{-}\boldsymbol{\pi})^2, \tag{17}$$

or the dual product

$$\pi * \pi = ({}^{+}\boldsymbol{\pi})^2 - ({}^{-}\boldsymbol{\pi})^2. \tag{18}$$

The tensor π being bivector, i.e. antisymmetrised tensor product of two vectors, is equivalent to the dual product vanishing. We see that this property holds in average (but the square of (18) has already nonzero VEV).

The main problem is whether description in terms of lengths can arise in the framework of the considered area formalism. Possible approach could be to treat unambiguity of the linklengths as a specific feature of the existing state of the Universe. Since the question is about unambiguity of the lengths in the full discrete spacetime, not only in it's 3D sections, a generalisation of the usual quantum mechanical notion of the state is implied. Generally the measure can be viewed as a linear functional $\mu(\Psi)$ on the (sub)space of functionals $\Psi(\{\pi\})$ on the superspace of area tensors π each point of which is represented by the set of the values of area tensors $\{\pi\}$ of all the triangles of the Regge manifold. Of interest is some hypersurface Γ in this superspace which consists of the points $\{\pi\}$ defining Regge manifolds with unambiguous lengths. Could we consistently define our measure on the subspace of the functionals of the form

$$\Psi(\{\pi\}) = \psi(\{\pi\})\delta_{\Gamma}(\{\pi\}) \tag{19}$$

where $\delta_{\Gamma}(\{\pi\})$ is (many-dimensional) δ-function with support on Γ? Further, such property as positivity is required to hold anyway to ensure probabilistic interpretation of the measure. Therefore a strict proof is required that the physically reasonable choice of tensors (given as parameters) n besides the trivial one $n = 0$ considered above exists which leaves the measure (13) positive.

The present work was supported in part by the Russian Foundation for Basic Research through Grant No. 01-02-16898, through Grant No. 00-15-96811 for Leading Scientific Schools and by the Ministry of Education Grant No. E00-3.3-148.

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